Prof. Dale van Harlingen, UIUC, Physics 498 Superconducting Quantum Devices

Lecture 3: Models and theories of superconductivity

Start discussion of how to understand the superconducting state

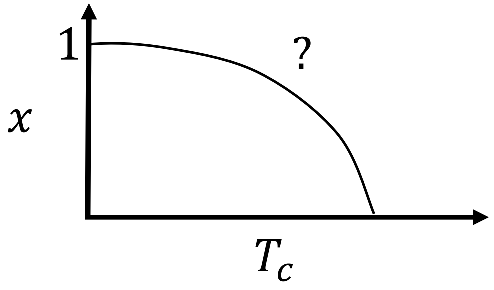
Present four phenomenological approaches:

1. Order parameter picture --- the Gorter-Casimir "two-fluid" model
2. Electrodynamics picture --- the London equations
3. Non-local electrodynamics model --- the Pippard extension to the London equations

<u>Two Fluid Model</u> Gorter – Casimir (1934) (analogous to that in superfluid He^4, but preceded it)

Thermodynamics + two fluids ⇒ T-dependences of measurable quantities: $G(T) \Rightarrow S(T)$, $C(T)$ and $H_c(T)$

★ Order parameter = fraction of SC electrons



$$G(T) = xGS + f(x)G_N$$

Minimize $G(T)$ to get $x(T)$

NORMAL CARRIERS: $G_N(T) = -\frac{1}{2}\gamma T^2$ from $S = \gamma T = -\frac{\partial G}{\partial T}$

SC CARRIERS: $G_S(T) = -\frac{1}{8\pi} H_c^2(T)$

1st guess: $G(T) = xG_S(T) + (1-x)G_N(T)$ Then, $\frac{dG}{dx} = 0 \Rightarrow G_N = GS \Rightarrow$ phase equilibrium

(No information about x)

2nd guess: $G(T) = xGS(T) + (1-x)^{1/2} G_N(T)$ Then, $\frac{dG}{dx} = 0 \Rightarrow x = 1 - \left(\frac{G_N}{2GS}\right)^2 = 1 - \frac{(2\pi\gamma)^2}{H_c^4} T^4$

Choose $T_c = \frac{H_c}{(2\pi\gamma)^2}$ so that $x(T_c) = 0$

Then, $$x(T) = 1 - \left(\frac{T}{T_c}\right)^4 \sim nS$$

density of superconducting electrons

Calculate quantities:

$$\Delta G = G_N - G_S = \frac{H_c}{8\pi} \quad \Rightarrow \quad H_c(T) = \left[8\pi\left(G_N - G_S\right)\right]^{1/2} \sim 1 - \left(\frac{T}{T_c}\right)^{1/2}$$

$$C_S(T) = -T\frac{d^2G_S}{dT^2} \quad \sim \quad T^3$$

Interesting model but not very justifiable

Significance:

- 1st use of concept of an order parameter
- Idea of a superfluid + normal excitations --- key concept of BCS and useful for transport and non-equilibrium effects
- Focus on thermal properties --- gets right form for electrodynamics , e.g. $\lambda(T)$

London Equations F. London, H. London (1935)

Focus on describing the electrodynamics
Variation of $\vec{J}$ with time, space (screening currents)

Return to the superconductor as a perfect conductor:

1. Forces:

$$\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = e\vec{E} \rightarrow \frac{d\vec{v}}{dt} = \frac{e}{m}\vec{E}$$

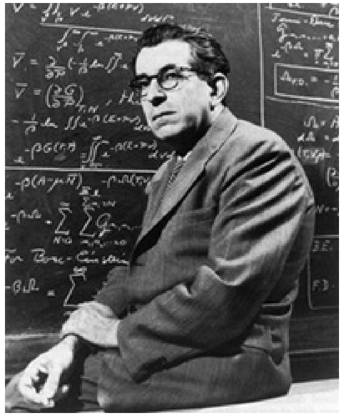
$$\vec{J} = ne\vec{v} \rightarrow \frac{d\vec{J}}{dt} = ne\frac{d\vec{v}}{dt} = \frac{ne^2}{m}\vec{E}$$

$$\left(\frac{m}{ne^2}\right)\frac{d\vec{J}}{dt} = \vec{E}$$

1st London Equation

$$\boxed{\Lambda\frac{d\vec{J}}{dt} = \vec{E}}$$

$$\Lambda = \frac{m}{ne^2} = \text{London Parameter}$$



Fritz London

Heinz London

The Electromagnetic Equations of the Supraconductor

By F. and H. London, Clarendon Laboratory, Oxford

(Communicated by F. A. Lindemann, F.R.S.—Received October 23, 1934)

Electric currents are commonly believed to persist in a supra-conductor without being maintained by an electromagnetic field. Thus the relation between the field strength **E** and the current density **J** in a supraconductor has sometimes been described† by means of an "acceleration equation," of the form

$$\Lambda\dot{\mathbf{J}} = \mathbf{E}\ ; \quad \Lambda = m/ne^2. \qquad (1)$$

This equation, which might replace Ohm's law for supraconductors, simply expresses the influence of the electric part of the Lorentz force on freely movable electrons of the mass m and charge e, the number per cm^3 being n (we use rational units). By definition the constant Λ must be positive. As a direct consequence of this equation (1) stationary currents in supraconductors are possible when $\mathbf{E} = 0$.

We shall see, however, that actually equation (1), which we will refer to as the "acceleration theory," implies more than is verified by experiment; moreover, presupposing an acceleration without any friction it implies a premature theory, the development of which has presented a hopelessly insoluble problem to mathematical physicists. Apparently a model was wanted which would explain that in its most stable state the supraconductor has always a persistent current. We shall give a formulation which is somewhat more restricted in this respect. On the other hand it includes one more important fact, namely, the experiment of Meissner and Ochsenfeld.‡ In this way we get a new description of the electromagnetic field in a supraconductor, which is consistent and, as it eliminates unnecessary statements, is in closer contact with experiment. This new description seems to provide an entirely new point of view for a theoretical explanation.

† Becker, Heller, and Sauter, 'Z. Physik,' vol. 85, p. 772 (1933); Braunbeck, 'Z. Physik,' vol. 87, p. 470 (1934); London, 'Nature,' vol. 133, p. 497 (1934).
‡ 'Naturw.,' vol. 21, p. 787 (1933).

2. Magnetic Fields:

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\, \vec{J}$$

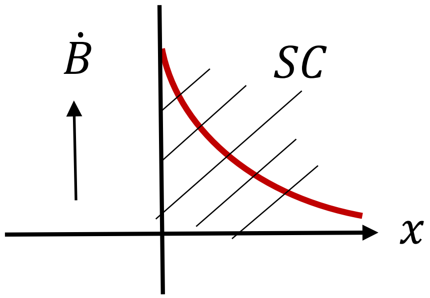
1st London Equation

$$\vec{\nabla} \times \dot{\vec{B}} = \frac{4\pi}{c}\, \dot{\vec{J}} = \frac{4\pi}{c\Lambda}\, \vec{E}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \dot{\vec{B}}\right) = \frac{4\pi}{c}\, \vec{\nabla} \times \dot{\vec{J}} = \frac{4\pi}{c\Lambda}\, \vec{\nabla} \times \vec{E} = \frac{4\pi}{c^2\Lambda}\, \dot{\vec{B}}$$

$$\Lambda\left(\vec{\nabla} \times \dot{\vec{J}}\right) = -\frac{1}{c}\, \dot{\vec{B}} \quad \Rightarrow \quad \nabla^2\, \dot{\vec{B}} = +\frac{4\pi}{c^2\Lambda}\, \dot{\vec{B}}$$

1-D geometry



$$\frac{d^2\dot{B}}{dx^2} = +\frac{4\pi}{c^2\Lambda}\, \dot{B}$$

$$\dot{B}(x) = \dot{B}(x)\, e^{-x/\lambda}$$

$$\lambda = \left(\frac{c^2\Lambda}{4\pi}\right)^{1/2}$$

$\dot{B}$ excluded from bulk (expected for perfect conductor)

★ Meissner effect ⇒ $\vec{B}$ excluded

2nd London Equation

$$\boxed{\Lambda\, \vec{\nabla} \times \vec{J} = -\frac{1}{c}\vec{B}}$$

LONDON EQUATIONS

cgs

$$\frac{d}{dt}(\Lambda\vec{J}\,) = \vec{E} \qquad \vec{\nabla}\times(\Lambda\vec{J}\,) = -\frac{1}{c}\vec{B} \qquad \Lambda = \frac{m}{ne^2}$$

MKS

$$\frac{d}{dt}\Lambda\vec{J} = \vec{E} \qquad \vec{\nabla}\times(\Lambda\vec{J}\,) = -\vec{B} \qquad \Lambda = \frac{m}{ne^2}$$

Can these be justified?

Start with the "canonical momentum" $\vec{p} = m\vec{v} + \frac{e}{c}\vec{A}$

We <u>expect</u> $\langle\vec{p}\rangle = 0$ in the ground state at zero field. <u>Assume</u> same at finite field.

$$\langle v\rangle = -\frac{e}{mc}\vec{A}$$

$$\vec{J} = ne\langle v\rangle = -\frac{ne^2}{mc}\vec{A}$$

$$\Lambda\vec{J} = -\frac{1}{c}\vec{A}$$

$$\frac{d}{dt}(\Lambda\vec{J}) = -\frac{1}{c}\frac{d\vec{A}}{dt} = \vec{E}$$

$$\vec{\nabla}\times(\Lambda\vec{J}) = -\frac{1}{c}\vec{\nabla}\times\vec{A} = -\frac{1}{c}\vec{B}$$

} London equations

Comments:

(a) $\vec{J} \propto \vec{A}$ not gauge invariant Valid only for London gauge:

$$\vec{\nabla} \cdot \vec{A} = 0 = \vec{\nabla} \cdot \vec{J} = 0 \qquad (J_\perp = 0 \text{ at surface})$$

(b) "Rigidity" of wavefunction $\leftrightarrow$ electrons not affected by a magnetic field

(c) $\langle \vec{p} \rangle = 0$ suggests "condensation" into $\vec{p} = 0$ state $\Rightarrow$ Bose pairing of electrons

Corrections:

Going back to the "two-fluid picture: London equation applies only to the superfluid part

$\vec{J} \longrightarrow \overrightarrow{J_s}$ superfluid part only Normal fluid: $\vec{J}_n = \sigma_n \vec{E}$ does not obey London

$\vec{E} \longrightarrow \vec{E}' = -\vec{\nabla}\phi = \vec{E} - \frac{1}{e}\vec{\nabla}\mu$ includes chemical potential

$n \longrightarrow n_s(T)$ superfluid density

Applications

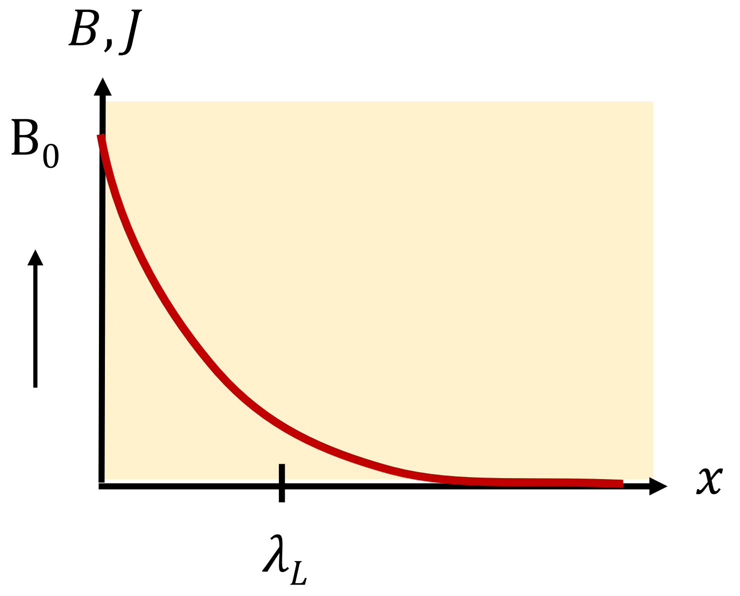
(1) Steady-state: $\vec{E}' = 0$

(2) Field expulsion: $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J}$

$$\nabla^2 \vec{B} = \left(\frac{4\pi}{\Lambda c^2}\right)\vec{B} = \frac{1}{\lambda_L}\vec{B}$$

$$B(x) = B_0\, e^{-x/\lambda_L}$$

$$J(x) = \underbrace{\frac{cB_0}{4\pi\lambda_L}}_{J_0}\, e^{-x/\lambda_L}$$



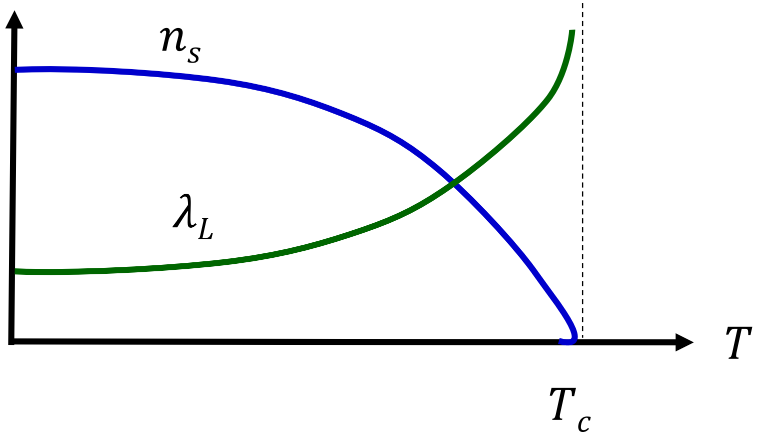
$$\lambda_L = \left(\frac{mc^2}{4\pi n s e^2}\right)^{1/2} = \left(\frac{m}{\mu_0 n_s e^2}\right)^{1/2}$$

cgs $\quad\quad$ MKS

Superconductor	λ_L
Al	50 nm
Pb	40 nm
Nb	85 nm
PbBi	200 nm
NbTi	300 nm
Nb_3Sn	65 nm
YBCO (a,b)	140 nm
YBCO (c)	700 nm

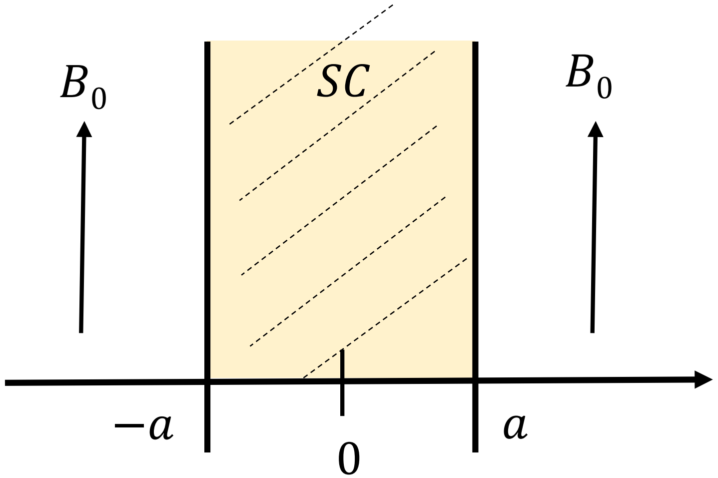
Temperature dependence:

$$\lambda_L(T) = \lambda_L(0)\left[1 - \left(\frac{T}{T_c}\right)^4\right]^{-1/2} \sim \frac{1}{n_s(T)}^{1/2}$$



Consistent w/ two-fluid model

(3) Field applied to a finite thickness plate



Width = $2a$

$$\frac{d^2B(x,\lambda_L)}{dx^2} = \frac{1}{\lambda_L{}^2} B(x)$$

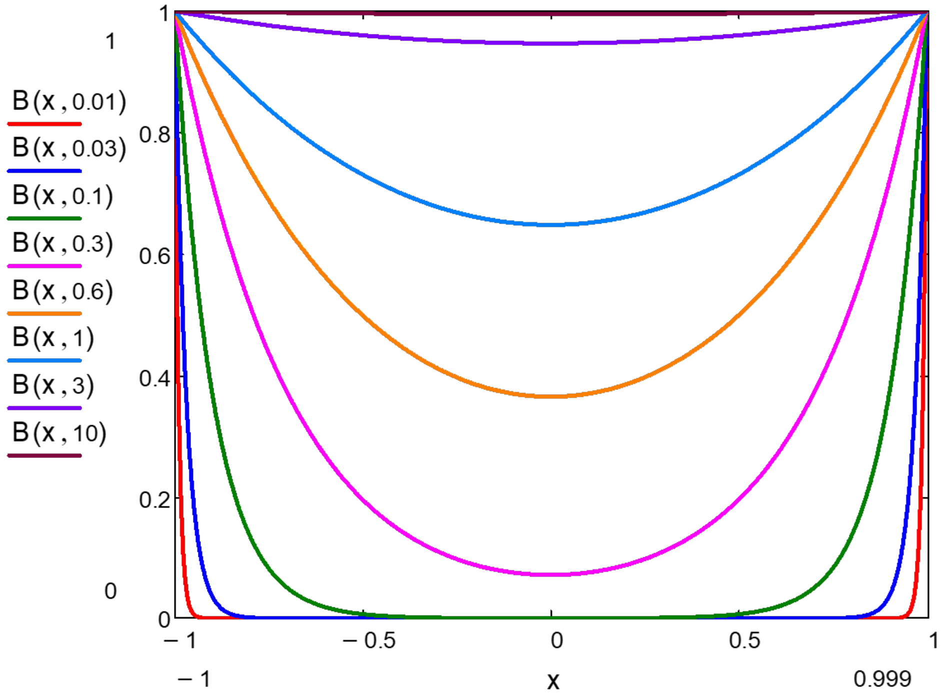
$$B(x,\lambda_L) = B_1 e^{-x/\lambda_L} + B_2 e^{+x/\lambda_L}$$

$$B.C. \quad B(\pm a) = B_0$$

$$\boxed{B(x,\lambda_L) = B_0 \frac{\cosh\left(\frac{x}{\lambda_L}\right)}{\cosh\left(\frac{a}{\lambda_L}\right)}}$$

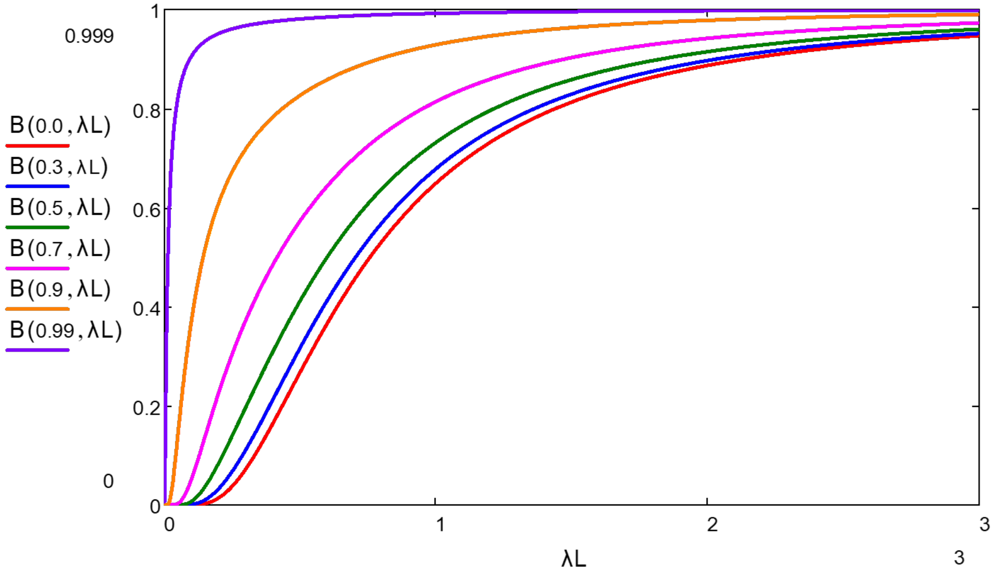
B vs. x

For different λ_L

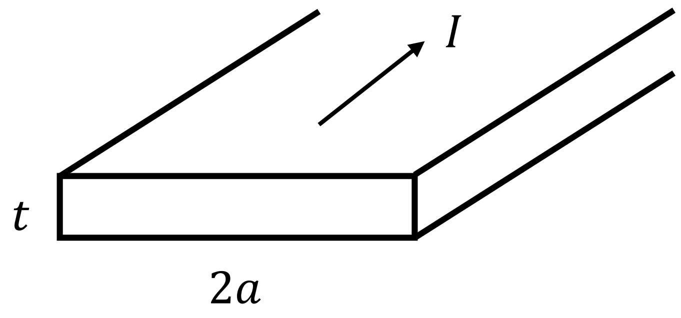


B vs. λ_L

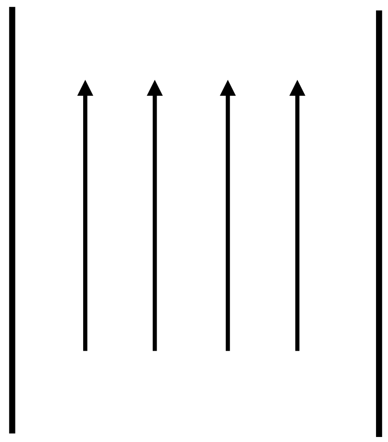
For different x

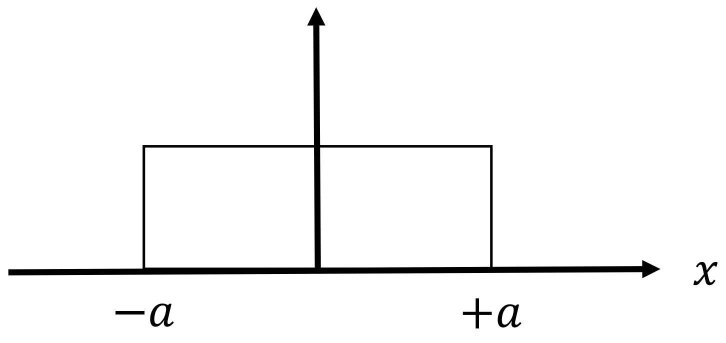


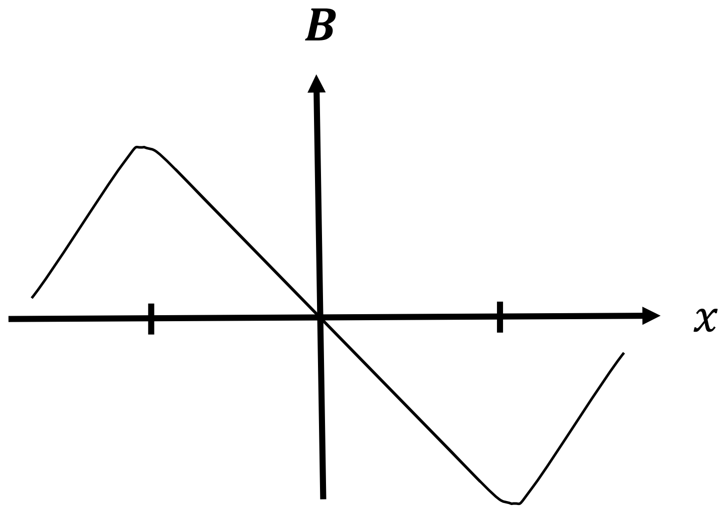
(4) Current flow in a strip



(a) Current density (uniform): $J = \dfrac{I}{2at}$ But this will be modified to prevent field penetration into the sample

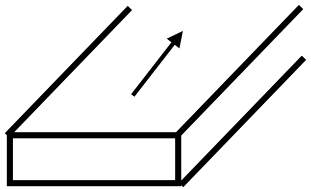






Self-fields from currents will be screened by SC

(b) Self-consistent London solution $(t \ll a)$



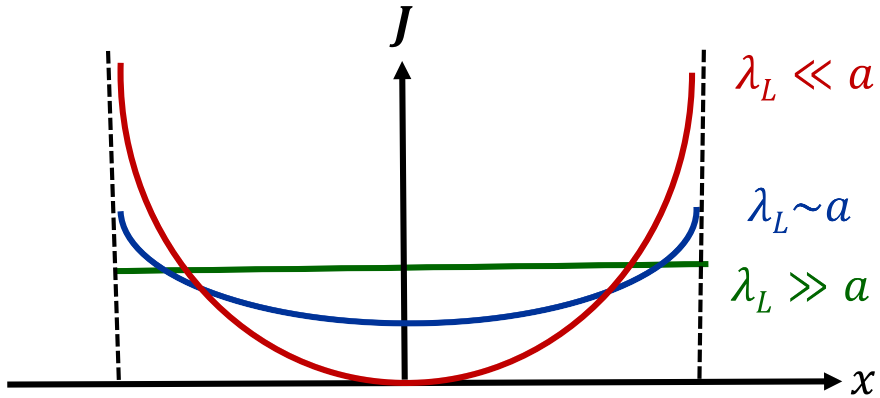
Solve for current flow $J(x)$:

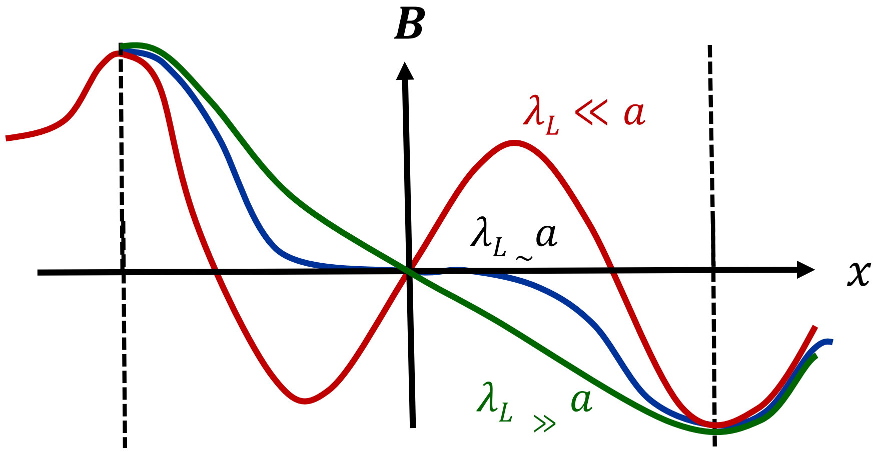
$$\nabla^2 J = \frac{1}{\lambda_L^2} J(x)$$

$$J(x) = A\left(e^{x/\lambda_L} + e^{-x/\lambda_L}\right)$$

$$I = \int_{-a}^{a} J(x)\ t dx$$ current conservation

$$J(x) = \left(\frac{I}{2xt}\right) \frac{\cosh\left({x}/{\lambda_L}\right)}{\sinh\left({a}/{\lambda_L}\right)}$$

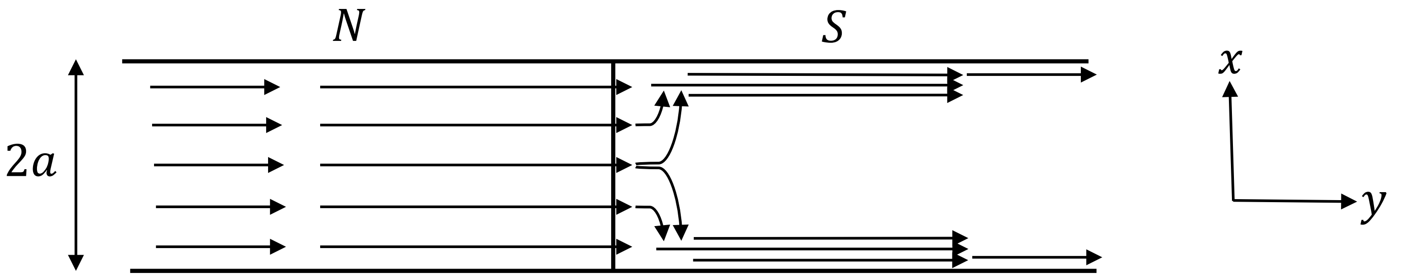




For $\lambda \gg a,$ $J(x) \rightarrow \frac{I}{2at}$ uniform

For $\lambda_L \ll a,$ $J(x) \rightarrow$ current piles up on edge

(5) Current flow at N-S interface



2-D problem inside S

$$\nabla^2 Jx = \frac{1}{{\lambda_L}^2}\ Jx \qquad \nabla^2 Jy = \frac{1}{{\lambda_L}^2}\ Jy$$

$$\vec{\Delta}\cdot\vec{J} = 0 = \frac{dJ_x}{dx} + \frac{dJ_y}{dy}$$

Boundary conditions:

$$J_x(y \to \infty) \to 0$$

$$J_y(y = 0) = JN$$

Current constraint:

$$I = \int_{-a}^{a} J_y(x)\ t\ dx = 2a\ t\ J$$

$$J_x(x,y) = -J\sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\ \frac{(a/\lambda_L)^2}{\sqrt{(a/\lambda_L)^2+(n\pi)^2}}\ \sin\frac{n\pi x}{a}\ \exp\left\{-\left[\left(\frac{1}{\lambda_L}\right)^2+\left(\frac{n\pi}{a}\right)^2\right]^{1/2} y\right\}$$

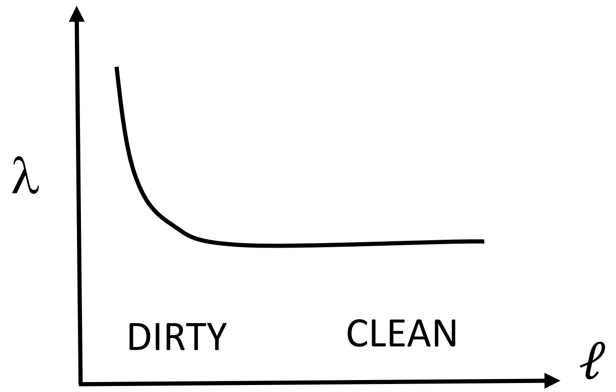
$$J_y(x,y) = -J\left[\frac{(a/\lambda_L)\cosh(x/\lambda_L)}{\sinh(a/\lambda_L)} - \sum_{n=1}^{\infty}\frac{2(-1)^n\ a/{\lambda_L}^2\cos\left(\frac{n\pi x}{a}\right)}{(a/\lambda_L)^2+(n\pi)^2}\ \exp\left\{-\left[\left(\frac{1}{\lambda_L}\right)^2+\left(\frac{n\pi}{a}\right)^2\right]^{1/2} y\right\}\right.$$

Beyond London ….. limitations of the London equations:

(1) Describe superfluid response – must put in normal response separately ⇒ two-fluid model

(2) Applies to weak magnetic fields ⇒ cannot handle vortices, intermediate state, inhomogeneous materials, …

(3) Predicts no dependence on impurities (contradicts experiment)



Determined from microwavwe wave absorbtion measurements of λ in doped samples:

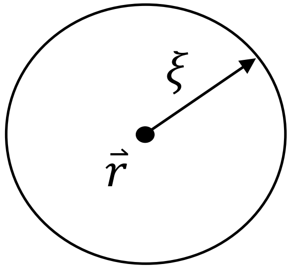
Sn + 3% In doubled λ

(4) Local equations: $\vec{J}(\vec{r}) = -\frac{1}{c\Lambda}\,\vec{A}(\bar{r})$

Pippard Non-local Modification to $\vec{J}(\vec{A})$

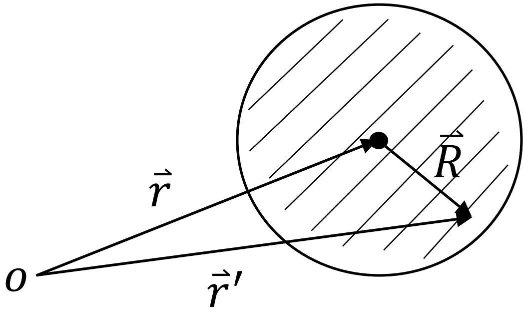
Brian Pippard (Cambridge, 1953)

$\vec{J}(\vec{r})$ depends on a weighted-average of $\vec{A}$ over a range ξ = coherence length



LONDON $$\vec{J}_s(\vec{r}) = -\frac{1}{c\Lambda}\vec{A}(\vec{r})$$

PIPPARD $$\vec{J}_s(\vec{r}) = -\frac{1}{c\Lambda}\left\{\frac{3}{4\pi\xi_0}\int\frac{\vec{R}\left[\vec{R}\cdot\vec{A}(\vec{r}')\right]}{R^4}\,\ell^{-R/\xi}\,d^3r'\right\}$$



$$\vec{R} = \vec{r}' - \vec{r}$$

How did he get this form? CHAMBERS expression for non-local resistivity

$$\vec{J}_n(\vec{r}) = \sigma\left\{\frac{3}{4\pi\ell}\int\frac{\vec{R}\left[\vec{R}\cdot\vec{E}(\vec{r}')\right]}{R^4}\,\ell^{-R/\ell}\,d^3r'\right\}$$

(replacing $\vec{J}_n(\vec{r}) = \sigma\vec{E}(\vec{r})$)

Here, range of influence is ℓ due to memory over time between scattering events

Range of non-locality for SC:

ξ = Pippard coherence length

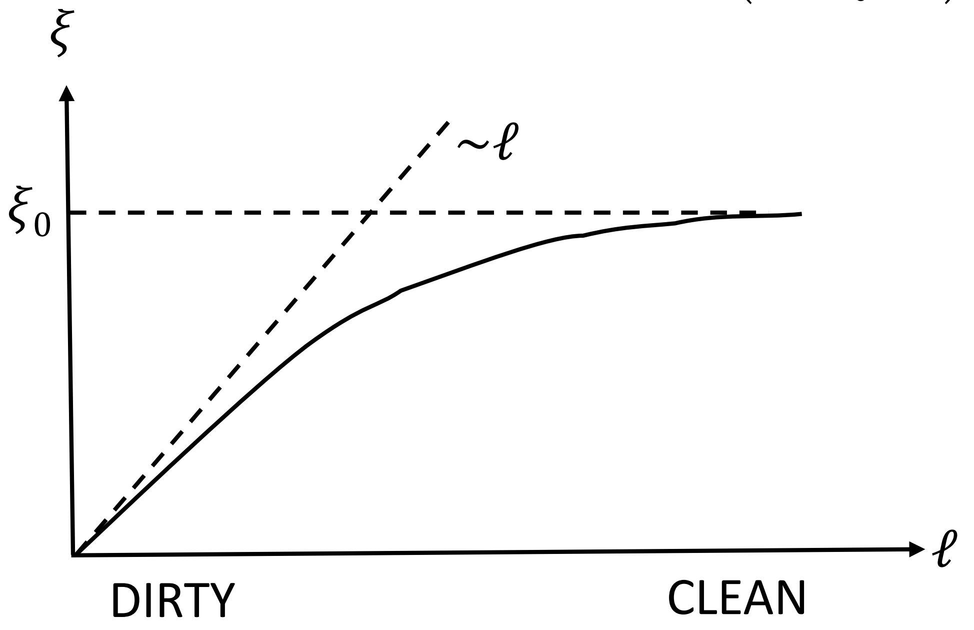
CLEAN SC ($\ell > \xi_0$)

$$\xi = \xi_0$$

DIRTY SC ($\ell \ll \xi_0$)

$$\xi(\ell) = \frac{\ell\,\xi_0}{\xi_0 + \ell} \sim \ell$$

$$\left(\frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{\ell}\right)$$

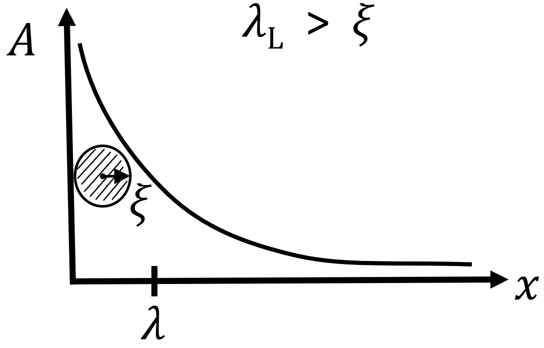


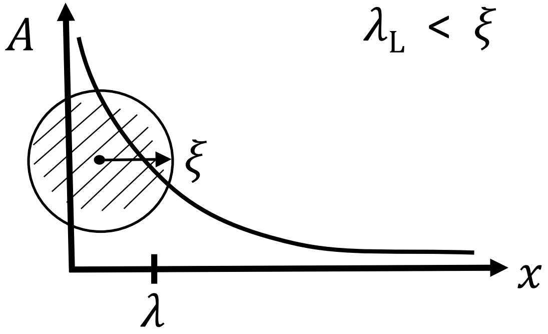
Values of ξ_0

Al	$1600\ nm$
Sn	$230\ nm$
Pb	$83\ nm$
Nb	$38\ nm$
$PbBi$	$20\ nm$
Nb_3Sn	$4\ nm$
$YBCO\ (a,b)$	$1.5\ nm$
$YBCO\ (c)$	$0.4\ nm$

Non-locality:

λ vs. ξ $\Rightarrow$ LOCAL (London) vs. NON-LOCAL (Pippard)





Averages current from higher A $\Rightarrow$ effective increase in J by $\left(\frac{\xi}{\lambda}\right)$

$$\vec{J} = -\frac{1}{c\Lambda}\vec{A}$$

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = -\frac{1}{c\Lambda}\bar{A}$$

$$\nabla^2 B = -\frac{4\pi}{c^2\Lambda}\vec{B} = -\left(\frac{1}{\lambda_L^2}\right)\vec{B}$$

$$\lambda = \lambda_L$$

$$\vec{J} \sim -\frac{1}{c\Lambda}\left(\frac{\lambda}{\xi}\right)\vec{A}$$

$$\vec{J} = \frac{c}{4\pi}\vec{\nabla}\times\vec{B} = -\frac{1}{c\Lambda}\left(\frac{\lambda}{\xi}\right)\bar{A}$$

$$\nabla^2 B = -\frac{4\pi}{c^2\Lambda}\left(\frac{\lambda}{\xi}\right)\vec{B} = -\left[\frac{1}{\lambda_L^2}\left(\frac{\lambda}{\xi}\right)\right]\vec{B} = -\left(\frac{1}{\lambda^2}\right)\vec{B}$$

$$\frac{1}{\lambda^2} = \frac{\lambda}{\lambda_L^2\xi}$$

$$\lambda = \left(\frac{\sqrt{3}}{2\pi}\right)^{1/\xi}\lambda_L\left(\frac{\xi}{\lambda_L}\right)^{1/\xi} > \lambda_L$$

~0.65

$$\lambda = (\lambda_L^2\xi)^{1/3} = \lambda_L\left(\frac{\xi}{\lambda_L}\right)^{1/3}$$

Non-locality over the coherence length modifies the penetration depth depending on the impurity concentration:

In CLEAN LIMIT, can be either London or Pippard limit:

LONDON $\lambda_L > \xi_0 \Rightarrow \lambda_P = \lambda_L$

PIPPARD $\lambda_L < \xi_0 \Rightarrow \lambda_P = \lambda_L \left(\frac{\xi_0}{\lambda_L}\right)^{1/3} > \lambda_L$

In moderately DIRTY limit $\lambda_P < \xi \sim \ell < \xi$

$$\lambda_P = \lambda_L \left(\frac{\ell}{\lambda_L}\right)^{1/3} \geq \lambda_L$$

In very DIRTY limit $\xi \sim \ell < \lambda_L, \xi_0$

Always in London (local) limit

Pippard expression: $\ell^{-R/\xi} \Rightarrow \ell^{-\frac{R}{\ell}}$

$$J \sim -\frac{1}{c\Lambda}\left(\frac{\ell}{\xi_0}\right)\vec{A} \Rightarrow \boxed{\lambda_P = \lambda_L \left(\frac{\xi_0}{\ell}\right)^{1/2} > \lambda_L}$$